Συμβαση Einstein

$$[C^{\mu\nu}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad [X_\mu] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad Y^\mu = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

A)

Εστω $B^\mu = C^{\mu\nu} X_\nu$

$$B^0 = C^{00}X_0 + C^{01}X_1 + C^{02}X_2 + C^{03}X_3$$
$$= 1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4 = 30$$

$$B^1 = C^{10}X_0 + C^{11}X_1 + C^{12}X_2 + C^{13}X_3$$
$$= 5\cdot 1 + 6\cdot 2 + 7\cdot 3 + 8\cdot 4 = 70$$

$$B^2 = C^{20}X_0 + C^{21}X_1 + C^{22}X_2 + C^{23}X_3$$
$$= 9\cdot 1 + 10\cdot 2 + 11\cdot 3 + 12\cdot 4 = 110$$

$$B^3 = C^{30}X_0 + C^{31}X_1 + C^{32}X_2 + C^{33}X_3$$
$$13\cdot 1 + 14\cdot 2 + 15\cdot 3 + 16\cdot 4 = 150$$

$$[C^{\mu\nu}X_\nu] = \begin{bmatrix} 30 \\ 70 \\ 110 \\ 150 \end{bmatrix}$$

Εστω $B^{\mu} = C^{\mu\nu} X_{\nu}$

$$B^{0} = C^{00} X_0 + C^{10} X_1 + C^{20} X_2 + C^{30} X_3$$
$$= 1 \cdot 1 + 5 \cdot 2 + 9 \cdot 3 + 13 \cdot 4 = 90$$

$$B^{1} = C^{01} X_0 + C^{11} X_1 + C^{21} X_2 + C^{31} X_3$$
$$= 2 \cdot 1 + 6 \cdot 2 + 10 \cdot 3 + 14 \cdot 4 = 100$$

$$B^{2} = C^{02} X_0 + C^{12} X_1 + C^{22} X_2 + C^{32} X_3$$
$$= 3 \cdot 1 + 7 \cdot 2 + 11 \cdot 3 + 15 \cdot 4 = 110$$

$$B^{3} = C^{03} X_0 + C^{01} X_1 + C^{02} X_2 + C^{03} X_3$$
$$4 \cdot 1 + 8 \cdot 2 + 12 \cdot 3 + 16 \cdot 4 = 120$$

$$\left[C^{\mu\nu} X_{\nu} \right] = \begin{bmatrix} 90 \\ 100 \\ 110 \\ 120 \end{bmatrix}$$

$$Y^{\mu} X_{\mu} = Y^0 X_0 + Y^1 X_1 + Y^2 \cdot X_2 + Y^3 X_3$$
$$5 \cdot 1 + 6 \cdot 2 + 7 \cdot 3 + 8 \cdot 4 = 70$$

$$X_{\mu} Y^{\mu} = X_0 Y^0 + X_1 Y^1 + X_2 Y^2 + X_3 Y^3 = 70$$

$$[x_\mu y^\nu] = \begin{bmatrix} 1\cdot 5 & 1\cdot 6 & 1\cdot 7 & 1\cdot 8 \\ 2\cdot 5 & 2\cdot 6 & 2\cdot 7 & 2\cdot 8 \\ 3\cdot 5 & 3\cdot 6 & 3\cdot 7 & 3\cdot 8 \\ 4\cdot 5 & 4\cdot 6 & 4\cdot 7 & 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 10 & 12 & 14 & 16 \\ 15 & 18 & 21 & 24 \\ 20 & 24 & 28 & 32 \end{bmatrix}$$

B)

$$[\eta_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & I_{3\times 3} \end{bmatrix} = \eta$$

$$\eta \cdot \eta^{-1} = I \rightarrow \eta^2 \cdot \eta^{-1} = \eta$$

$$\eta^2 = \begin{bmatrix} -1 & 0 \\ 0 & I_{3\times 3} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & I_{3\times 3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{3\times 3} \end{bmatrix} = I$$

apa $\eta^2 \cdot \eta^{-1} = \eta \rightarrow I \cdot \eta^{-1} = \eta \rightarrow \boxed{\eta^{-1} = \eta}$

$$[x^\mu] = [\eta^{\mu\nu} x_\nu]$$

$$x^0 = \overset{-1}{\eta^{00}} x_0 + \cancel{\eta^{01} x_1} + \cancel{\eta^{02} x_2} + \cancel{\eta^{03} x_3} = -x_0 = -1$$

$$x^1 = \cancel{\eta^{10} x_0} + \overset{1}{\eta^{11}} x_1 + \cancel{\eta^{12} x_2} + \cancel{\eta^{13} x_3} = x_1 = 2$$

$$x^2 = \cancel{\eta^{20} x_0} + \cancel{\eta^{21} x_1} + \overset{1}{\eta^{22}} x_2 + \cancel{\eta^{23} x_3} = x_2 = 3$$

$$x^3 = \cancel{\eta^{30} x_0} + \cancel{\eta^{31} x_1} + \cancel{\eta^{32} x_2} + \overset{1}{\eta^{33}} x_3 = x_3 = 4$$

$$[x^\mu] = [\eta^{\mu\nu} x_\nu] = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$[y_\mu] = [\eta_{\mu\nu} y^\nu] = \begin{bmatrix} -5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

$$[C_\mu{}^\nu] = [\eta_{\mu\kappa} C^{\kappa\nu}]$$

$$C_0{}^0 = \overset{-1}{\eta_{00}} C^{00} + \cancel{\eta_{01}} C^{10} + \cancel{\eta_{02}} C^{20} + \cancel{\eta_{03}} C^{30} = -1$$

$$C_0{}^1 = \overset{-1}{\eta_{00}} C^{01} + \cancel{\eta_{01}} C^{11} + \cancel{\eta_{02}} C^{21} + \cancel{\eta_{03}} C^{31} = -2$$

$$C_0{}^2 = \overset{-1}{\eta_{00}} C^{02} + \cancel{\eta_{01}} C^{12} + \cancel{\eta_{02}} C^{22} + \cancel{\eta_{03}} C^{32} = -3$$

$$C_0{}^3 = \overset{-1}{\eta_{00}} C^{03} + \cancel{\eta_{01}} C^{13} + \cancel{\eta_{02}} C^{23} + \cancel{\eta_{03}} C^{33} = -4$$

$$C_1{}^0 = \cancel{\eta_{10}} C^{00} + \overset{1}{\eta_{11}} C^{10} + \cancel{\eta_{12}} C^{20} + \cancel{\eta_{13}} C^{31} = 5$$

$$C_2{}^0 = \cancel{\eta_{20}} C^{00} + \cancel{\eta_{21}} C^{10} + \overset{1}{\eta_{22}} C^{20} + \cancel{\eta_{23}} C^{30} = 9$$

$$C_3{}^0 = \cancel{\eta_{30}} C^{00} + \cancel{\eta_{31}} C^{10} + \cancel{\eta_{32}} C^{20} + \overset{1}{\eta_{33}} C^{30} = 13$$

$$C_1{}^1 = \cancel{\eta_{10}} C^{0} + \overset{1}{\eta_{11}} C^{11} + \cancel{\eta_{12}} C^{21} + \cancel{\eta_{13}} C^{31} = 6$$

$$c_1{}^2 = \cancel{\eta_{10}}\, c^{02} + \overset{1}{\eta_{11}}\, c^{12} + \cancel{\eta_{12}}\, c^{22} + \cancel{\eta_{13}}\, c^{32} = 7$$

$$c_1{}^3 = \cancel{\eta_{10}}\, c^{03} + \overset{1}{\eta_{11}}\, c^{13} + \cancel{\eta_{12}}\, c^{23} + \cancel{\eta_{13}}\, c^{33} = 8$$

ομοιως τα υπολοιπα στοιχεια $c_i{}^j = \eta_{ik} c^{kj} = \delta_{ik} c^{kj} = c^{ij}$

$$[c_\mu{}^\nu] = [\eta_{\mu\kappa} c^{\kappa\nu}] = \begin{bmatrix} -1 & -2 & -3 & -4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \eta \cdot C$$

$$[c^\mu{}_\nu] = [c^{\mu\kappa} \eta_{\kappa\nu}] = C \cdot \eta$$

για συντομια η λυση σε μορφη γινομενο πινακων

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 2 & 3 & 4 \\ -5 & 6 & 7 & 8 \\ -9 & 10 & 11 & 12 \\ -13 & 14 & 15 & 16 \end{bmatrix} = [c^\mu{}_\nu]$$

$$[C_{\mu\nu}] = [\eta_{\mu\sigma} C^{\sigma\kappa} \eta_{\kappa\nu}] = \eta \cdot C \cdot \eta$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 & 4 \\ -5 & 6 & 7 & 8 \\ -9 & 10 & 11 & 12 \\ -13 & 14 & 15 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -3 & -4 \\ -5 & 6 & 7 & 8 \\ -9 & 10 & 11 & 12 \\ -13 & 14 & 15 & 16 \end{bmatrix} = [C_{\mu\nu}]$$